

A Comparison of the Proper Time Equation and the Renormalization Group β -Function in String Theory

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Abstract

It is known that there is a proportionality factor relating the β -function and the equations of motion viz. the Zamolodchikov metric. Usually this factor has to be obtained by other methods. The proper time equation, on the other hand, is the full equation of motion. We explain the reasons for this and illustrate it by calculating corrections to Maxwell's equation. The corrections are calculated to cubic order in the field strength, but are exact to all orders in derivatives. We also test the gauge covariance of the proper time method by calculating higher (covariant) derivative corrections to the Yang-Mills equation.

1 Introduction

The β -function or renormalization group approach provides a convenient way of obtaining the low energy equations of string theory. [1, 2, 3, 4, 5]. There have been many generalizations of this method subsequently, to massive modes [6, 7, 8, 10, 11], to higher order corrections in gauge theories [12, 13, 14, 15], to connections with Wilson's renormalization group equations [8, 9, 10], to connections with string field theory [16, 17, 18], to background independent formulations [19, 20, 21] and in other ways. The proper-time approach [10] is a generalization of a technique that has been used for point particles[23, 24, 25, 26, 27, 28] and is a variant of the β -function approach that has some advantages. One of the advantages is that by virtue of its similarity with an S-matrix calculation it is guaranteed to reproduce the full equations of motion. The β -function, on the other hand, is not identical to the equation of motion - it is only *proportional* to the equation of motion. This was shown in the case of the tachyon [10].

More recently this method has been generalized to include gauge fields [22, 29], both, Abelian and non-Abelian. In the Abelian case we derived a string generalization of the covariant Klein Gordon equation for scalar particles coupled to electromagnetism. In the non-Abelian case we derived the Yang Mills equation. Some time ago it was shown that string corrections to Maxwell action gives the Born-Infeld action in a low energy limit [13, 14]. This was done by calculating the partition function in the presence of background electromagnetic fields. Subsequently it was shown that the same result could be obtained in the conventional β -function approach [12]. However it was found that the β -function is not quite equal to the equations of motion, rather it was proportional to it[12]. The prefactor was fixed by requiring that the equations come from an action. This proportionality factor is the Zamolodchikov metric[31] as was shown in ref[30]. Polyakov showed that the equations of motion are related to the β function as follows:

$$\frac{\delta \mathcal{L}}{\delta \phi^i} = G_{ij} \beta^j \quad (1.1)$$

Here G_{ij} is the Zamolodchikov metric. This was later worked out for the case of the tachyon [10]. As mentioned earlier, it was also shown there that the proper time equation gives the full equation, prefactor and all. It is a natural question, then, as to whether one can get the full corrections (including the

Zamolodchikov metric) to the Maxwell equation using the proper time approach. In this paper we address this question and show that the proper time equation does give the full equation of motion. We do this in two different ways. The first method is a straightforward generalization of [10] and uses on-shell gauge fixed vertex operators. This calculation is very similar to an S-matrix calculation. Thus we obtain the leading (non-trivial) term of the Born-Infeld action. But in addition it has all the momentum dependence included, i.e., all the higher derivative corrections to the low energy equations of motion are included. In this sense it goes beyond the conventional β -function calculation. The second method uses the gauge covariant vertex operators introduced in [29] and gives manifestly gauge covariant equations. The calculation is a little more tedious than the first method, but it, too, gives the entire momentum dependence.

The gauge covariant method has been used to derive the Yang-Mills equations [29]. It is appropriate then to ask whether it can be used to calculate higher order corrections just as in the Abelian case. As a test of the method we calculate higher derivative corrections to Yang Mills equation and check that the result is covariant.

This paper is organized as follows: In Section II we use the gauge fixed version of the proper time equation to calculate the leading corrections to Maxwell's equation and explain the difference between this calculation and a β -function calculation. In Sec III we do this in the gauge covariant method. In Section IV we summarize the results of a calculation of higher order corrections to the Yang-Mills equation. We conclude in Section V with some comments.

2 Gauge Fixed Proper Time Equation

The proper time equation is

$$\frac{d}{d\ln z} z^2 < ik_1^\mu \partial_z X^\mu e^{ik_0 X} i l_1^\nu \partial_w X^\nu e^{il_0 \cdot X(0)} > = 0 \quad (2.1)$$

If we extract the coefficient of k_1^μ in the above equation we get $\frac{\delta \mathcal{L}}{\delta A_\mu}$. The expectation value uses the measure

$$\int \mathcal{D}X \exp\{i \int_M d^2 z \partial_z X^\mu \partial_{\bar{z}} X^\mu + \int_{\partial M} A^\mu(X) \partial_z X^\mu dz\} \quad (2.2)$$

Here $A_\mu(X) = \int dk_0 A_\mu(k_0) e^{ik_0 X}$ is the background Abelian gauge field. We have used the notation $A^\mu(k_0) \equiv k_1^\mu$; $A^\rho(q_0) \equiv q_1^\rho \dots etc.$ here and below.

The first non-zero correction to Maxwell's equation is due to two insertions of the A_μ vertex operator. Thus we have to calculate:

$$\begin{aligned} & \frac{d}{d\ln z} z^2 \int_0^z du \int_0^u dv \\ & < ik_1^\mu \partial_z X^\mu e^{ik_0 \cdot X(z)} i p_1^\rho \partial_u X^\rho e^{ip_0 \cdot X(u)} i q_1^\sigma \partial_v X^\sigma e^{iq_0 \cdot X(v)} i l_1^\nu \partial_w X^\nu e^{il_0 \cdot X(0)} > \\ & = 0 \end{aligned} \quad (2.3)$$

We have chosen a particular ordering of momenta here. In an S-matrix calculation we would have to sum over all possible orderings, but here, since the momentum is an integration variable, we do not have to worry about that. The vertex operators are assumed to satisfy the physical state conditions, namely,

$$k_0^2 = k_0 \cdot k_1 = 0 \quad (2.4)$$

There are various possible contractions that can be made in (2.3). We will concentrate on those that give rise to terms of the form $(k_1 \cdot p_1 q_1 \cdot l_1)$, $(k_1 \cdot q_1 p_1 \cdot l_1)$ or $(k_1 \cdot l_1 p_1 \cdot q_1)$.

Thus, consider the first one, viz., $k_1 \cdot p_1 q_1 \cdot l_1$:

$$\begin{aligned} & \frac{d}{d\ln z} z^2 \int_0^z du \int_0^u dv \\ & \frac{1}{(z-u)^2} \frac{1}{v^2} (z-u)^{k_0 \cdot p_0} (z-v)^{k_0 \cdot q_0} (z)^{k_0 \cdot l_0} (u-v)^{p_0 \cdot q_0} (u)^{p_0 \cdot l_0} (v)^{q_0 \cdot l_0} \end{aligned} \quad (2.5)$$

The integral becomes:

$$(z)^{k_0.l_0} \int_0^z du (z-u)^{k_0.p_0-2} (u)^{p_0.l_0+q_0.l_0+p_0.q_0-1} \quad (2.6)$$

$$\int_0^1 dv' (z-v)^{k_0.q_0} (1-v')^{p_0.q_0} (v')^{q_0.l_0-2}$$

Here $v' \equiv v/u$. We can expand $(z-v)^{k_0.q_0}$ in a powers of $k_0.q_0$ (if it is small). The lowest term is just 1, and we get:

$$(z)^{-2} \int_0^1 du' (1-u')^{k_0.p_0-2} (u')^{-1+\delta} \int_0^1 dv' (1-v')^{p_0.q_0} (v')^{q_0.l_0-2} \quad (2.7)$$

We have used $p.l + q.l + p.q \equiv \delta$ (≈ 0 for on-shell photons) and $u' \equiv \frac{u}{z}$. (We have dropped the subscript ‘0’ on the momentum variables in some of the equations.) Naively, acting on (2.7) with $\frac{d}{d \ln z} z^2$ would give zero since there is no z -dependence in the integrals. However, as shown in [10] one has to regulate the integrals in (2.5) by modifying the limits to $\int_a^{z-a} \int_a^{u-a}$ where a is a short distance cutoff on the world sheet. Thus (2.7) becomes:

$$\int_a^{1-\frac{a}{z}} du' (1-u')^{k_0.p_0-2} (u')^{-1+\delta} \int_{\frac{a}{u}}^{1-\frac{a}{u}} dv' (1-v')^{p_0.q_0} (v')^{q_0.l_0-2} \quad (2.8)$$

If we assume that $p.q \approx 0$, we need not regulate the upper limit of the integral. The second integral can be expanded in a power series in a/u to give:

$$B(-1 + q.l, p.q + 1) - \frac{(a/u)^{q.l-1}}{q.l-1} - \frac{(a/u)^{q.l}}{q.l} p.q + \dots \quad (2.9)$$

In (2.9) we have kept only the terms that diverge as $a \rightarrow 0$. The linear divergence is due to tachyon exchange and we subtract it by hand. If we insert (2.9) into (2.8) and do the u' -integral we get for the z -dependent part

$$\frac{(a/z)^{k.p-1}}{k.p-1} + \frac{(a/z)^{k.p}}{k.p} + \frac{(a/z)^\delta}{\delta} \quad (2.10)$$

and it multiplies

$$B(-1 + q.l, p.q + 1) - \frac{p.q}{q.l} \quad (2.11)$$

This is nothing but the S -matrix with its (massless) poles subtracted. Expanding (2.10) in powers of $k.p \approx \delta \approx 0$, we see that the coefficient of $\ln z$

is 2. Thus, as argued in [10], the proper-time equation is guaranteed to give the effective equations as determined from the S -matrix. Of course in the case of photon-photon scattering there are no massless poles in any channel since there is no three- photon vertex. In fact, we have verified explicitly that when contributions from all the terms involving permutations of the momenta are added, the poles cancel. In the case of Yang-Mills theory the poles will survive in the S -matrix, but will still be subtracted out in the proper-time equation, exactly as in the present case, (2.11). If we expand (2.11) we get

$$\frac{1}{q.l-1}[\frac{p.q}{q.l} + 1][1 - (q.l)(p.q)\zeta(2)] - \frac{p.q}{q.l} + \text{higher order...} \quad (2.12)$$

($\zeta(2) = \frac{\pi^2}{6}$) For $q.l \approx p.q \approx 0$ we thus get

$$-1 - 2(k.q)(p.q)\zeta(2) \quad (2.13)$$

Thus, finally, the contribution to the equation is a term of the form:

$$k_1.p_1 q_1.l_1(k.q)(p.q) = \frac{k_1.p_1 q_1.l_1}{2}[(l.q)^2 - (p.q)^2 - (k.q)^2] \quad (2.14)$$

which comes from a term

$$[F^4 - 1/4(F^2)^2] \quad (2.15)$$

which is precisely the quartic piece of the Born-Infeld action. Note that the -1 in (2.13) did not get subtracted because it did not come from a logarithmic divergence. Nevertheless when all the permutations are added it drops out, just as in the S -matrix calculation. (Of course, we must remember that the proper-time equation is an equation of motion. Thus the contribution to the equation of motion that we have obtained is really the part that multiplies k_1^μ in (2.14). This obviously corresponds, upto an overall combinatoric factor, to varying w.r.t A_μ in (2.15).)

Let us now turn to the β -function calculation of [12]. The cubic term in the β -function is

$$(F^2)_{\lambda\nu}\partial^\nu F_\mu^\lambda \quad (2.16)$$

It can easily be checked that it differs from the full equation (at this order) by terms proportional to $\partial_\mu F_{\mu\nu}$, i.e. Maxwell's equation. Thus, in particular, the $(F^2)^2$ term (more precisely, its variation) is not contained in the β -function.

It is easy to see why. In a β -function calculation, Maxwell's equation is the coefficient of the logarithmic divergence in the lowest order graph. In a higher order graph, a lower order divergence is necessarily cancelled by a counter-term in any renormalization scheme. Thus at the cubic order the β -function will not have anything proportional to Maxwell's equation. The proper-time equation, on the other hand, picks out a logarithmic divergence.¹ - it does not matter whether it is from a divergent subgraph or an overall divergence. Thus terms proportional to $\partial_\mu F^{\mu\nu}$ would show up in a higher order calculation. The only subtractions are $(\log)^2$ (or higher) divergences - which corresponds to subtracting pole terms multiplying the $\ln z$ piece. This is exactly the procedure for determining the effective action from an S -matrix. The above arguments thus show why the β function cannot be equal to the equation of motion, and also why it is plausible that the proper-time equation is the same as the equation of motion. In the case of the tachyon it was shown in [10] that the two are indeed equivalent to all orders. In the present case we have verified it only to cubic order (in the field strength, but to all orders in derivatives). However, the nature of the argument in [10] did not depend in any specific way on the properties of the tachyon and therefore we expect it to go through for all the modes.

Finally, it is important to note that (2.11) has the exact momentum dependence (except for the restriction $k.q \approx 0$). Thus, this method gives *all the higher derivative corrections* to the low energy equations of motion. This is very difficult to do in a β -function calculation.

¹It actually looks for $\ln z$ pieces. But $\ln z$ always occurs as $\ln(z/a)$, where a is the short distance cutoff, for dimensional reasons, so the two procedures are equivalent.

3 Gauge Covariant Proper Time Equation

We now repeat this calculation in a manifestly covariant way using the techniques of [29]. We use the following identity derived in [29]:

$$e^{ik_0 X} ik_1 \partial X = \int_0^1 d\alpha \partial(e^{i\alpha k_0 X} ik_1 X) + \int_0^1 d\alpha [e^{i\alpha k_0 X} X^\nu \partial X^\mu (i)^2 \alpha k_0^{[\nu} k_1^{\mu]}] \quad (3.1)$$

and insert in the proper time equation used there:

$$\frac{d}{d\Sigma} \int dz_1 \int dz_2 < ik_1^\mu \partial_{z_1} X^\mu e^{ik_0 X} i l_1^\nu \partial_{z_2} X^\nu e^{il_0 X} > \quad (3.2)$$

As explained in [29, 22], the integrals over z_1, z_2 allow us to throw away total divergences and thereby makes the result gauge invariant. This calculation is similar to that in [29] where we calculated the cubic and quartic terms in the Yang-Mills coupling. Comparing with that calculation, it is easy to see that only the second term on the RHS of (3.1) contributes in the Abelian case.² Thus to lowest order we just have Maxwell's equation:

$$\int dz_1 \int dz_2 \int d\alpha \alpha \int d\beta \beta k_0^{[\mu} k_1^{\nu]} l_0^{[\rho} l_1^{\sigma]} < X^\mu \partial_{z_1} X^\nu e^{i\alpha k_0 X} X^\rho \partial_{z_2} X^\sigma e^{i\beta l_0 X} > \quad (3.3)$$

$$= 1/4 \int dz_1 \int dz_2 k_0^{[\mu} k_1^{\nu]} l_0^{[\rho} l_1^{\sigma]} (\delta^{\mu\rho} \delta^{\nu\sigma} \Sigma \partial_{z_1} \partial_{z_2} \Sigma + \delta^{\mu\sigma} \delta^{\nu\rho} \partial_{z_1} \Sigma \partial_{z_2} \Sigma) \quad (3.4)$$

where $\Sigma = \ln(z_1 - z_2)$. If we now follow the procedure of [29] and vary w.r.t. Σ , we get

$$1/2 \int dz_1 \int dz_2 k_0^{[\mu} k_1^{\nu]} l_0^{[\rho} l_1^{\sigma]} (\delta^{\mu\rho} \delta^{\nu\sigma} - \delta^{\mu\sigma} \delta^{\nu\rho}) \partial_{z_1} \partial_{z_2} \Sigma = 0 \quad (3.5)$$

The coefficient of k_1^μ is Maxwell's equation: $\partial_\mu F^{\mu\nu} = 0$.

One can also follow the technique [10] of looking at the logarithmic deviation from $\frac{1}{(z_1 - z_2)^2}$. Thus we operate on (3.5) with $\frac{d}{d\ln(z_1 - z_2)} (z_1 - z_2)^2$. In this case we do not integrate over z_1, z_2 . We then get the same result:

$$1/4 k_0^{[\mu} k_1^{\nu]} l_0^{[\rho} l_1^{\sigma]} \frac{d}{d\ln(z_1 - z_2)} (\ln(z_1 - z_2) + 1) \quad (3.6)$$

$$= 1/4 k_0^{[\mu} k_1^{\nu]} l_0^{[\rho} l_1^{\sigma]} \quad (3.7)$$

²The other terms contribute to the cubic and quartic Yang-Mills coupling.

The coefficient of k_1^μ gives Maxwell's equation as before, except that the coefficient is $1/4$ rather than 1 .³

We now look at the next non-trivial order - namely quartic, since the cubic term is easily shown to vanish. Our aim is to show explicitly that the proper-time equation produces the contribution of the $(F^2)^2$ term in the Born-Infeld action. To this end we consider the following:

$$\begin{aligned} & \int dz_2 \int dz_3 \int d\alpha d\beta d\gamma d\delta \\ & < X^{\mu_1} \partial_{z_1} X^{\nu_1} e^{i\alpha k_0 X} X^{\mu_2} \partial_{z_2} X^{\nu_2} e^{i\beta p_0 X} X^{\mu_3} \partial_{z_3} X^{\nu_3} e^{i\gamma k_0 X} X^{\mu_4} \partial_{z_4} X^{\nu_4} e^{i\delta p_0 X} > \\ & k_0^{[\mu_1} k_1^{\nu_1]} p_0^{[\mu_2} p_1^{\nu_2]} \end{aligned} \quad (3.8)$$

We consider a specific contraction of the type $\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}\delta^{\mu_3\mu_4}\delta^{\nu_3\nu_4}$. We get the proper time equation:

$$\begin{aligned} & \frac{d}{d\ln(z_1)} (z_1)^2 F^2 F^2 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \\ & \left[\frac{\ln(z_1 - z_2) + 1}{(z_1 - z_2)^2} \right] \left[\frac{\ln z_3 + 1}{z_3^2} \right] \\ & (z_1 - z_2)^{\alpha\beta k_0 p_0} (z_1 - z_3)^{\alpha\gamma k_0 q_0} (z_2 - z_3)^{\beta\gamma p_0 p_0} (z_2 - z_4)^{\beta\delta p_0 l_0} (z_3 - z_4)^{\gamma\delta q_0 l_0} (z_1)^{\alpha\delta k_0 l_0} \\ & = 0 \end{aligned} \quad (3.9)$$

There is a new momentum conservation equation:

$$\alpha k_0 + \beta p_0 + \gamma q_0 + \delta l_0 = 0 \quad (3.10)$$

that constrains $\alpha, \beta, \gamma, \delta$. For simplicity we will refer to αk_0 as k etc. and restore these factors at the end. We rewrite (3.9) as:

$$\frac{d}{d\ln z_1} z_1^2 z_1^{k.l} \int dz_2 (z_1 - z_2)^{-2+k.p+\nu} z_2^{p.l} \int dz_3 z_3^{-2+q.l+\mu} (z_2 - z_3)^{p.q} (z_2 - z_3)^{k.q} = 0 \quad (3.11)$$

If we expand (3.11) in powers of ν, μ , they multiply powers of $\ln(z_1 - z_2)$ and $\ln z_3$ respectively. Thus we evaluate (3.11) for general μ, ν , expand the

³This discrepancy is not important for the purposes of this paper. However it is an issue that needs to be resolved.

result in powers of μ, ν , and look at the μ, ν - independent terms, those linear in μ, ν and those bilinear in μ, ν . The sum of these terms (with μ, ν set to 1) will give us the result of the integral (3.9). To simplify the integral we set $k.q = 0$. The integral gives:

$$z_1^{\mu+\nu} B(-1 + k.p + \nu, \mu) B(-1 + q.l + \mu, 1 + p.q) \quad (3.12)$$

We should actually regularize the integral, but since it just has the effect of subtracting the poles, we will do it by hand at the end. Using the expansion

$$\frac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(1+x+y)} = 1 - xy\zeta(2) + \dots$$

we get for the coefficient of $\ln z_1$:

$$\begin{aligned} & (\mu + \nu) \left\{ \left(1 + \frac{\mu}{k.p + \nu - 1}\right) \left(1 + \frac{\mu}{k.p + \nu}\right) \frac{1}{\mu} [1 - \mu(k.p + \nu)\zeta(2)] \right\} \\ & \left\{ \frac{1}{\mu + q.l - 1} \left[1 + \frac{p.q}{q.l + \mu}\right] [1 - (q.l + \mu)p.q\zeta(2)] \right\} \end{aligned} \quad (3.13)$$

The pole terms are to be subtracted as usual. If we look at the term multiplying $\zeta(2)$ we find:

$$\int d\alpha \alpha \int d\beta \beta \int d\gamma \gamma \int d\delta \delta \left[-\frac{\mu\nu}{q.l - 1} \zeta(2) \right] = \frac{1}{16} \mu\nu \zeta(2) + \text{higher order in } q.l \quad (3.14)$$

Thus (3.9) becomes $\frac{1}{16} \mu\nu \zeta(2) F^2 F^2$. Thus we see that there is an $F^2 F^2$ with some non-zero coefficient, which is what we wanted to show.

This calculation is a little more tedious than the β -function calculation. But as mentioned in the previous section, we should realize that we have a result that is *valid to all orders in momenta*. All we have to do is to expand the Beta function in (3.12) in powers of momenta and do the $\alpha, \beta, \gamma, \delta$ integrals (note that k in (3.12) stands for αk_0 and so on). The only approximation that has been made is that we have set $k.q = 0$, thus our result is correct to lowest order in $k.q$.

4 Higher Order Corrections in Yang-Mills Theories

As another application of the gauge covariant proper-time formalism of [10, 22, 29] we have calculated higher derivative corrections to the Yang-Mills equation - corrections of the form $Tr F^3$ or $DFDF$ [15]. More precisely, we have calculated the terms involving three A fields and three derivatives (The term involving four derivatives and two A fields is trivial to calculate), and verified explicitly that the result is covariant. The calculation is straightforward, but a little tedious, and we will only give an outline and some intermediate results. Our main purpose is to test the method.

We start with the following:

$$Tr[T^a T^b T^c] \int dz \int_w^z dz_1 \int dw < ik_1^a \partial_z X e^{ik_0 X} i q_1^b \partial_{z_1} X e^{iq_0 X} i p_1^c \partial_w X e^{ip_0 X(w)} > \quad (4.1)$$

We replace each vertex operator with the expression on the RHS of (3.1), reproduced here for convenience:

$$e^{ik_0 X} ik_1 \partial X = \underbrace{\int_0^1 d\alpha \partial(e^{i\alpha k_0 X} ik_1 X)}_a + \underbrace{\int_0^1 d\alpha [e^{i\alpha k_0 X} X^\nu \partial X^\mu (i)^2 \alpha k_0^{[\nu} k_1^{\mu]}]}_b \quad (4.2)$$

We will refer to the total derivative piece as ‘a’ and the “gauge invariant” piece as ‘b’. Substituting (4.2) into (4.1) gives eight terms that we can conveniently label as : ‘aaa’, ‘aba’, ‘aab’ ... etc. The antisymmetry of the terms will ensure that $Tr[T^a T^b T^c]$ is multiplied by an antisymmetric (a, b, c) term, thus converting it to f^{abc} . The ‘aaa’ term is easily seen to be zero. The ‘aab’ term gives (suppressing group theory indices)

$$- < e^{i\alpha(k_0+k_1)X(z)} \partial_z e^{i\beta(q_0+q_1)X(z)} i^2 \gamma^2 X^\sigma \partial_w X^\theta p_0^{[\sigma} p_1^{\theta]} e^{i\gamma p_0 X(w)} > \quad (4.3)$$

We have written k_1 and q_1 in the exponent with the understanding that we are to keep only the piece linear in $k_1 q_1$. This gives, for the piece cubic in momentum,

$$\int dw \int dz \gamma^2 \alpha \beta p_0^{[\sigma} p_1^{\theta]} \{ [(k_0+k_1)^\sigma (q_0+q_1)^\theta \Sigma \partial_z \partial_w \Sigma \Sigma (-1) [\alpha(k_0+k_1) + \beta(q_0+q_1)] \cdot \gamma p_0] \} \quad (4.4)$$

$$+[(k_0 + k_1)^\theta (q_0 + q_1)^\sigma \partial_z \Sigma \partial_w \Sigma \Sigma (-1) [\alpha(k_0 + k_1) + \beta(q_0 + q_1)] \cdot \gamma p_0] \}$$

Here, as always $\Sigma = \ln(z - w)$. If we multiply by $(z - w)^2$ and pick the coefficient of $\ln(z - w)$ we are left with the second term $(\partial_w \Sigma \partial_z \Sigma \Sigma)$. On the other hand, treating Σ as a field [22, 29], if we allow integration by parts, we get contribution from both terms. Thus depending on which procedure we use, we get different numerical coefficients. This is the same ambiguity encountered in Sec III. Fortunately, all terms at this order have the same form, so we can consistently pick one procedure. Presumably, comparison with higher and lower order terms will allow us to decide which is the right procedure. For this calculation we allow ourselves to integrate by parts to bring everything into the form $\Sigma \partial_z \partial_w \Sigma$. This gives, after doing the α, β, γ integrals: (for ‘aab’)

$$1/4 p_0^{[\sigma} p_1^{\theta]} \{ k_1^{[\sigma} q_1^{\theta]} (k_0 + q_0) \cdot p_0 + k_0^{[\sigma} q_1^{\theta]} k_1 \cdot p_0 + k_1^{[\sigma} q_0^{\theta]} q_1 p_0 \} \quad (4.5)$$

The other terms ‘aba’ and ‘baa’ are obtained by cyclic permutations. A similar calculation gives for the ‘abb’ term

$$- 3/4 k_1 \cdot (q_0 - p_0) q_0^{[\mu} q_1^{\nu]} p_0^{[\mu} p_1^{\nu]} \quad (4.6)$$

Adding (4.5) and (4.6) gives (we have suppressed the group indices of k, q, p above)

$$\{ \partial_\sigma (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\sigma [A_\mu, A_\nu] + \partial_\sigma (\partial_\mu A_\nu - \partial_\nu A_\mu) [A^\sigma, (\partial_\mu A_\nu - \partial_\nu A_\mu)] \} \quad (4.7)$$

with $A_\mu = A_\mu^a T^a$ and $tr\{T^a, [T^a, T^c]\} = f^{abc}$. (4.7) is obviously the cubic piece in $Tr\{D_\sigma F^{\mu\nu} D^\sigma F_{\mu\nu}\}$. Finally the ‘bbb’ term gives, upto an overall constant, $Tr[F^3]$. This verifies the gauge covariance of the corrections to Yang-Mills’ equations.

It would be interesting to see whether the concept of “covariant derivative” can be fruitfully introduced in such calculations to simplify the algebra.

5 Conclusions

We have compared the proper-time formalism with the β -function method. We have seen that the proper-time formalism gives the full equation of motion, including the prefactor, known usually as the Zamolodchikov metric. In the case of Abelian gauge fields we showed explicitly that the $(F^2)^2$ term is obtained. We did this in two different ways - one being gauge fixed and very similar to the S -matrix calculation. The other method is gauge covariant. It is important to point out that in both cases one can get results to arbitrary high order in momenta with very little work. The reason it is easier than calculating the β -function is that one does not have to worry about subtracting lower order divergences. It may in fact turn out that an easier way to calculate the β -function is to calculate the proper time equation and divide by the Zamolodchikov metric.

In Section IV we also presented results regarding higher order (derivative) corrections in Yang-Mills theory using the covariant proper-time method. We thus verified that the method is consistent and gives gauge covariant results.

In conclusion, the proper time method seems promising as a means of calculating low energy equations of motion in a gauge covariant way. It can also be extended off-shell as in [29] by introducing a finite world sheet cutoff. The massive modes also appear in a natural way there. We hope that by considering both the low energy and high energy systems by means of the same equation one can interpolate smoothly between them. This should provide some insight into the symmetries of strings and the role of the massive modes.

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